Name of College - S.S. College, J-Bad

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B.Sc Part III Deptt - Maths Time - 10 A.M to 11 AM

Date - 26-05-2020

TOPIC - Defferentiation Under the Sign of Integration

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Introduction: → The method, differentiation under the sign of integration is applied to integrating functions which are not easily integrable. This method can be applied to definite as well as indefinite integrals provided the limits of integration are independent of the quantity with respect to which differentiation is done.

Problem: →

$$\int_0^\infty \frac{\tan^{-1}\alpha x \, \tan^{-1}\beta x}{x^2}\,dx = \frac{\pi}{2}\log\left\{\frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^\alpha \beta^\beta}\right\}$$

Solution: →

$$\text{Let } I = \int_0^\infty \frac{\tan^{-1}\alpha x \, \tan^{-1}\beta x}{x^2}\,dx$$

Diff. both sides with respect to α, we get →

$$\frac{dI}{d\alpha} = \int_0^\infty \frac{x}{(1+\alpha^2 x^2)} \frac{\tan^{-1}\beta x}{x^2}\,dx$$

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$$\Rightarrow \frac{dI}{d\alpha} = \int_0^\infty \frac{\tan^{-1}\beta x}{x(1+\alpha^2x^2)}\,dx$$

Diff. again with respect to β, we get—

$$\frac{d^2I}{d\beta\,d\alpha} = \int_0^\infty \frac{1}{x(1+\alpha^2x^2)}\cdot\frac{x}{(1+\beta^2x^2)}$$

$$= \int_0^\infty \frac{dx}{(1+\alpha^2x^2)(1+\beta^2x^2)}$$

Now $\frac{1}{(1+\alpha^2x^2)(1+\beta^2x^2)} = \frac{1}{\alpha^2-\beta^2}\left[\frac{\alpha^2}{1+\alpha^2x^2} - \frac{\beta^2}{1+\beta^2x^2}\right]$

Therefore,

$$\frac{d^2I}{d\beta\,d\alpha} = \frac{1}{(\alpha^2-\beta^2)}\int_0^\infty\left[\frac{\alpha^2}{1+\alpha^2x^2} - \frac{\beta^2}{1+\beta^2x^2}\right]dx$$

$$= \frac{1}{\alpha^2-\beta^2}\Big[\alpha\tan^{-1}\alpha x - \beta\tan^{-1}\beta x\Big]_0^\infty$$

$$= \frac{1}{\alpha^2-\beta^2}\left[\alpha\cdot\pi/2 - \beta\cdot\pi/2\right]$$

$$= \frac{1}{\alpha^2-\beta^2}(\alpha-\beta)\,\pi/2$$

$$= \frac{1}{\alpha+\beta}\cdot\pi/2$$

Now integrating with respect to β we get—,

$$\frac{dI}{d\alpha} = \pi/2\,\log(\alpha+\beta) + A$$

Where A is constant of integration independent of B.

When $\beta = 0$ $\frac{dI}{d\alpha} = 0$

Therefore,

$$0 = \frac{\pi}{2}\log\alpha + A$$

$$\Rightarrow A = -\pi/_2 \log\alpha$$

Therefore, we get–

$$\frac{dI}{d\alpha} = \frac{\pi}{2}\left[\log(\alpha+\beta) - \log\alpha\right]$$

Now integrating with respect to α, we get–

$$I = \frac{\pi}{2}\left[\int\log(\alpha+\beta)\,d\alpha - \int\log\alpha\,d\alpha\right]$$

Integrating by parts

$$I = \frac{\pi}{2}\left[\alpha\log(\alpha+\beta) - \int\frac{\alpha}{\alpha+\beta}\,d\alpha - \left\{\alpha\log\alpha - \int\alpha\cdot\frac{1}{\alpha}\,d\alpha\right\}\right] + B$$

When $\alpha = 0$ $I = 0$ therefore,

$$0 = \frac{\pi}{2}\,\beta\log\beta + B$$

$$\Rightarrow B = -\frac{\pi}{2}\,\beta\log\beta$$

Therefore

$$I = \frac{\pi}{2}\left[(\alpha+\beta)\log(\alpha+\beta) - \alpha\log\alpha - \beta\log\beta\right]$$

$$= \frac{\pi}{2}\left[\log(\alpha+\beta)^{\alpha+\beta} - \log\alpha^{\alpha} - \log\beta^{\beta}\right]$$

$$= \frac{\pi}{2}\left[\log(\alpha+\beta)^{\alpha+\beta} - \log\alpha^{\alpha}\beta^{\beta}\right]$$

$$= \frac{\pi}{2}\;\frac{\log(\alpha+\beta)^{\alpha+\beta}}{\log\alpha^{\alpha}\beta^{\beta}}$$, Hence the result.

Problem 2.

Evaluate

$$\int_0^\infty \frac{\tan^{-1}\alpha x}{x(1+x^2)}\,dx$$ where $\alpha > 0$

Solution :→ Let

$$I = \int_0^\infty \frac{\tan^{-1}\alpha x}{x(1+x^2)}\,dx \quad \text{——} \quad (1)$$

Diff. with respect to α, we get

$$\frac{dI}{d\alpha} = \int_0^\infty \frac{dx}{(1+\alpha^2x^2)(1+x^2)}$$

But $$\frac{1}{(1+\alpha^2x^2)(1+x^2)} = \frac{1}{(a^2-1)}\left[\frac{a^2}{1+a^2x^2} - \frac{1}{1+x^2}\right]$$

Therefore,

$$\frac{dI}{da} = \frac{1}{a^2-1}\left[\int_0^\infty \frac{a^2}{1+a^2x^2}\,dx - \int_0^\infty \frac{dx}{1+x^2}\right]$$

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$$\frac{dI}{d\alpha} = \frac{1}{a^2-1}\left[\alpha \tan^{-1}\alpha x - \tan^{-1}x\right]_0^{\infty}$$

$$= \frac{1}{a^2-1}\left[a\frac{\pi}{2} - \pi/2\right] = \frac{\pi(a-1)}{2(a^2-1)}$$

But $a > 1$

$$\therefore \frac{dI}{d\alpha} = \frac{a-1}{a^2-1}\pi/2$$

Integrating

$$I = \frac{\pi}{2}\int \frac{1}{a+1}\,da = \frac{\pi}{2}\left[\log(a+1) + \log A\right] \quad (2)$$

Putting $a=0$ in (1) and (2) we get-

$I = 0$ and $\log A = 0$

Therefore, from (2), we get-

$$I = \frac{\pi}{2}\log(1+a)$$